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Back of the Envelope Insights into Shower Energy Measurements by Sparse Ground Arrays

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Abstract

The energies of the highest energy cosmic rays are measured by sparse ground arrays with a precision of $20 \sim 30\%$. Analytical expressions, for electro-magnetic shower longitudinal and transverse distributions, provide a basis for *back of the envelope* estimates of various properties of large air showers.

This note:

- compares a simple analytical model to Akeno/AGASA data, and
- uses the model to provide insights into the uncertainties in the measurement of cosmic ray shower energies using sparse ground arrays.

1. Introduction

The energies of the highest energy cosmic rays are measured with an absolute precision of $20 \sim 30\%$ [1] by sparse ground arrays. This level of precision was possible because of an interesting observation by Hillas et al [2]. Hillas noted that a measurement of the track density several hundred meters from the shower core of extensive air showers was potentially less sensitive to variations in the depth of the initial interaction and/or in the primary cosmic ray composition than a measurement of the total number of charged particles, N_e. Additionally the track density, either $\rho(600)$ measured with water Cherenkov detectors at 600m or $S(600)$ measured with (thin) scintillators, was to a good approximation linearly related to the cosmic ray shower energy. Thus the affordable way to measure extensive air showers using sparse ground arrays was also the most precise!

The Hillas technique is typically validated by complex Monte Carlo simulations. These simulations are also used to estimate the *calibration* constant relating $\rho(600)$ or $S(600)$ to the cosmic ray shower energy, E_{shower} [2,3].

For purely electro-magnetic showers analytical expressions have been derived for the shower longitudinal and transverse distributions. Extensive air showers become dominantly electro-magnetic at some point in their development. Thus to what extent can large air showers be described by the analytical expressions for a purely electro-magnetic shower? In this note we *assume* that near and past shower maximum an extensive air shower can be approximated by an electro-magnetic shower:

- with the shower maximum, X_{max} in gm/cm^2, given by the observed average shower X_{max} versus energy [4,5] and
- with the number of charged particles at shower maximum, N_e^{max}, given by the observed average shower N_e^{max} versus energy [6,7].

The resulting simple analytical model is then:

- compared to Akeno/AGASA data, and
- used to provide insights into the uncertainties in the measurement of cosmic ray shower energies using sparse ground arrays.

2. Analytic shower model and comparison to Akeno/AGASA Results

2.1 Analytic shower model

Analytic formulas for electro-magnetic showers exist in the literature. The forms used in this analysis are the following:

- Longitudinal shower profile: The longitudinal shower profile for an electro-magnetic shower is a function of the effective depth in radiation lengths, T_{eff}, and/or of the related age parameter, s [8,9]:

  $$N_e(T_{eff}) \;=\; (\frac{0.31}{\sqrt{y}}) \cdot e^{T_{eff}\cdot(\;1.-1.5ln(s)\;)}$$

  where:

  - N_e is the number of charged particles,
  - $T_{eff} = T_{eff}^{max} + \Delta T$ is the effective depth in radiation lengths,
  - $\Delta T = (X_{expt} - X_{max})/36.7((gm/cm^2)/radiation\ length)$,
  - $y = ln\frac{E_{shower}}{E_{critical}} = T_{eff}^{max}$, and
  - the shower age is:

    $$s \;=\; \frac{3}{(1 + 2 \cdot (T_{eff}^{max}/T_{eff}))}$$

  For showers in the energy range $10^{17}eV \leq E_{shower} \leq 10^{20}eV$, the depths of shower maximum [4,5,12] are approximated by:

  $$X_{max} \;\approx\; 620 \;+\; \frac{200}{3} \cdot log_{10}(E_{shower}/10^{17}eV) \;\; (gm/cm^2)$$

  This is plotted as the solid line in Fig. 1.

  At shower maximum:

  $$N_e^{max} \;=\; (\frac{0.31}{\sqrt{y}}) \cdot e^{T_{eff}^{max}}$$

  This relation is used to evaluate T_{eff}^{max} with N_e^{max} estimated using the approximation

[6,7] [1]:

$$N_e^{max} \approx \frac{E_{shower}(eV)}{1.5 \times 10^9 (eV/e)}$$

- *Transverse shower profile*: The transverse shower profile for an electro-magnetic shower is also a function of the shower age parameter, s [10,11]:

$$\rho(r) = \frac{N_e(T_{eff})}{r_m^2} \cdot (\frac{r}{r_m})^{s-2} \cdot (1 + \frac{r}{r_m})^{s-4.5} \cdot \frac{\Gamma(4.5-s)}{2\pi\Gamma(s)\Gamma(4.5-2s)}$$

  where $\rho(r)$ is the charged particle (electron) track density/m^2 and r_m is the Moliere radius (in meters) in air $\sim$ 2 radiation lengths above the experiment.

- *Zenith angle variations*: For modest values of shower zenith angle the dominant variation with zenith angle is the change in depth, $X_{expt}(\theta)$, where the ground array samples the shower:

$$X_{expt}(\theta) = X_{atmosphere} \cdot sec(\theta)$$

  where $X_{atmosphere} = X_{expt}(0)$ is the depth for vertical incidence.

2.2 Akeno/AGASA Data Parameterization

The Akeno/AGASA experiment uses the charged particle track density, measured in a large array of plastic scintillators, to determine the location, direction and energy of extensive air showers. To determine the shower energy the following procedure is used. First the scintillator pulse heights are compared to an energy independent, empirical transverse (*i.e.* lateral) distribution function [13]:

$$\rho(r) = C \cdot (\frac{r}{r_m})^{-1.2} \cdot (1 + \frac{r}{r_m})^{-(\eta-1.2)} \cdot (1 + (\frac{r}{1000m})^2)^{-0.6}$$

where: $\eta = 3.97 - 1.79 \cdot (sec\theta - 1)$, $r_m = 91.6$m and C is a normalization factor obtained in the fitting procedure (along with the core location of the shower). The fit parameters are

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[1] Large showers are approximately Gaussian. Thus:

$$E_{shower} \approx N_e^{max} \cdot \sqrt{2\pi} \cdot \sigma_{shower} \cdot < dE/dx >$$

where $\sigma_{shower} \approx 230 \pm 10$ gm/cm^2 and $< dE/dx > \approx 2.34$ MeV/(gm/cm^2). If $\sim$ 10% of the shower energy is (essentially) unmeasured (ν's, μ's, ..), then $E_{shower} \approx 1.48 \times 10^9 eV \cdot N_e^{max}$.

then used to evaluate the track density $S_\theta(600) = \rho(600)$. For showers at non-zero zenith angles the measured track density is then corrected to 0° [13]:

$$S_0(600) \quad = \quad S_\theta \cdot e^{\left(\frac{X_{atmosphere}}{500}\cdot(sec\theta-1) \; + \; \frac{X_{atmosphere}}{594}\cdot(sec\theta-1)^2\right)}$$

where $X_{atmosphere} = 920\text{gm/cm}^2$ at Akeno. Finally the track density is related to the primary shower energy [3] by:

$$E_{shower} \quad = \quad (\ 2.03 \pm 0.10 \times 10^{17} eV/(\#/m^2)\ ) \cdot S_0^{1.02\pm0.02}(600)$$

For this study we use the form linear in S_0 [13]:

$$E_{shower} \quad = \quad 2.0 \times 10^{17} eV/(\#/m^2)\ ) \cdot S_0(600)$$

2.3 Comparison of analytic model to Akeno/AGASA data

To compare the electro-magnetic analytic model with the Akeno/AGASA data, we use only the functional parameterizations and approximations listed above. Note that several of the parameters have uncertainties at the $10 \sim 20\%$ level. The goal of this note is to check for areas of general agreement or disagreement between the analytic model and the Akeno/AGASA data:

- *Transverse particle density distribution*: The analytic model for the particle density versus distance from the shower core and the Akeno/AGASA parameterizations of their data are totally specified by the relations above and by the shower energy. Both models (parameterizations) should be appropriate for $10^{17}eV \sim E_{shower} \sim 10^{20}eV$. However, because of the power law decrease in the cosmic ray flux with energy the Akeno/AGASA parameterization is optimized at $\sim 10^{18}$ eV.

  A ratio of the Akeno transverse particle density distribution divided by the analytic model is plotted versus the radial distance from the shower core in Fig. 2. This ratio is within 20% of unity for $500m < r < 1000m$ over the energy range $10^{17}eV \sim E_{shower} \sim 10^{20}eV$. Thus the agreement in shape and in normalization is rather good.

  The analytic model transverse particle density distributions, scaled in shower energy to 10^{19}eV, are given in Fig. 3. These curves exhibit the feature that the transverse particle density distributions are more (less) peaked at higher (lower) shower energies, and that the relative difference between the curves is least for shower radii $\sim 600m$.

- *Zenith angle variations*: The Akeno/AGASA $S_\theta(600)$ data for non-zero zenith angles are corrected to $S_0(600)$ using the empirical relation given above. In the analytic model the effect of non-zero zenith angle is to change the effective depth in the shower where the ground array measures the particle density, *i.e.* $S_\theta(600)$. The analytic model and Akeno/AGASA parameterizations are compared in Fig. 4, where the analytic model predictions for $S_\theta(600)/E_{shower}$ are plotted versus the secant of the shower zenith angle. For the Akeno/AGASA data the equivalent form is:

$$\frac{S_\theta(600)}{E_{shower}} = \frac{1.}{2.0 \times 10^{17} eV/(\#/m^2)} \cdot e^{\left(\frac{-X_{atmosphere}}{500} \cdot (sec\theta - 1) \;+\; \frac{-X_{atmosphere}}{594} \cdot (sec\theta - 1)^2\right)}$$

This is shown by the solid line in Fig. 4. The agreement, particularly for $10^{18} eV \sim E_{shower} \sim 10^{19} eV$, is quite good.

3. *Back of the envelope* estimates and insights from the analytic shower model

The comparisons of the analytic shower model and Akeno/AGASA data suggest that the analytic model can provide *back of the envelope* estimates and insights into the measurement of the energy (and related properties) of extensive air showers. This echos Hillas' original observation [2] that $\rho(600)$ or $S(600)$ should be used to characterize the shower energy and that other event features (X_{max} or ρ_μ or ...) should be used to discriminate on intial cosmic ray composition.

Some interesting issues related to the measurement of shower energy include the following:

- *Why is $S(600)$ better than N_e?*: The actual question is "why does the charged particle density at $\sim$ 600m provide a more precise measure of the shower energy than a measurement of the number of charged particles, N_e"? As reviewed by Gaisser [9], the dominant source of fluctuations in the shower energy measurement by large ground arrays comes from variations in the depth of the original interaction in the atmosphere. This depth varies because of cross sections differences, for different compositions in the initial cosmic rays, as well as from the simple probabilistic nature of the depth of the first interaction. These fluctuations are minimized at and near the the *peak* in the shower longitudinal distribution.

  Fig. 5 and 6 show the longitudinal profiles for the number of charged particles, N_e, and for $S(600)$ respectively for showers in the range $10^{17} eV \sim E_{shower} \sim 10^{20} eV$.

The longitudinal profile for N_e peaks at depths significantly less than the depth of the Akeno/AGASA array. In contrast the longitudinal profile for $S(600)$ is near maximum for Akeno/AGASA vertical showers. The best match for vertical showers occurs for $E_{shower} \sim 10^{18}$ eV. For showers at nonzero zenith angles the $S(600)$ measurement is less optimal but improves as the shower energies approach 10^{20} eV. Thus in this simple model the fluctuations in $S(600)$ are predicted to be less than the fluctuations in N_e because the longitudinal profile for $S(600)$ is near maximum at the depth of the Akeno/AGASA array. Finally for showers at large zenith angles, with $X_{expt}(\theta)$ significantly greater than $X_{expt}(0)$, then $S(600)$ may be far from maximum. In these cases fluctuations in shower depth will result in fluctuations in the reconstructed shower energies. This is qualitatively consistent with the Akeno/AGASA estimates for $\Delta S(600)/S(600)$ [13].

- *How does the longitudinal profile of $S(r)$ vary with r?*: The analytic model predictions for the longitudinal dependence of the charged particle density, $S(r)$, at different radial distances from the shower core is shown in Fig. 7. These curves exhibit the general trend that track densities at greater distances from the core peak at greater depths in the shower. Furthermore the longitudinal profiles for the charged particle density from the analytic model are qualitatively similar to the Gaisser - Hillas form [14] for the total number of charged particles. The Fly's Eye group find that the Gaisser - Hillas shower parameterization provides a good description of their fluorescence events [5].

- *How does the analytic profile compare with Gaisser-Hillas?*: The longitudinal profile for the total number of charged particles, N_e, is shown for the Gaisser-Hillas parameterization and for the analytic model in Fig. 8. The analytic model curves at 10^{19}eV also include examples where the value of the effective number radiation lengths at shower maximum, T_{eff}^{max}, was changed by ±2 from the nominal value. The analytic model is in good agreement with the Gaisser-Hillas parameterization near and past shower maximum. Past shower maximum the two parameterizations agree to better than 10% until $N_e/N_e^{max} \leq 0.2$

- *How do the radial distributions for particle density vary with zenith angle?*: Parameterizations of the radial distribution of particle densities typically include a correction for zenith angle; see Akeno/AGASA form above. The analytic model prediction is shown in Fig. 9 for showers at 10^{19}eV. The radial distributions, $S(r)$, at different zenith angles are normalized to agree at $r = 600$m using the results from Fig. 4. The normalization factor differs from one by $< 20\%$ for zenith angles $\leq 30°$ consistent with the longitudinal dependences of $S(600)$ in Fig. 6. The curves show the qualitative feature of being *flatter* for showers at large zenith angles.

4. Summary

The analytic formulas for electro-magnetic showers were observed to provide a good description of the Akeno/AGASA scintillator based ground array shower energy measurements near and past shower maximum. These analytic formulas were then used to obtain *back of the envelope* insights into why the charged particle track density, $S(600)$, at 600m from the shower core provides a measurement of cosmic ray shower energies having the minimum fluctuations. Finally the conventional ground array zenith angle correction was found to be consistent with a simple correction for the change in depth, $X_{expt}(\theta)$, where the ground array samples the shower.

Acknowledgements and Apologies

I would like to thank Clem Pryke, for numerous interesting observations, and Mike Hillas and John Linsley, for their many insightful papers, all of which peaked my curiosity to try to understand large air showers (at least at some level). I would like to apologize to those I did not reference, who already know all of this and who have probably written this before.

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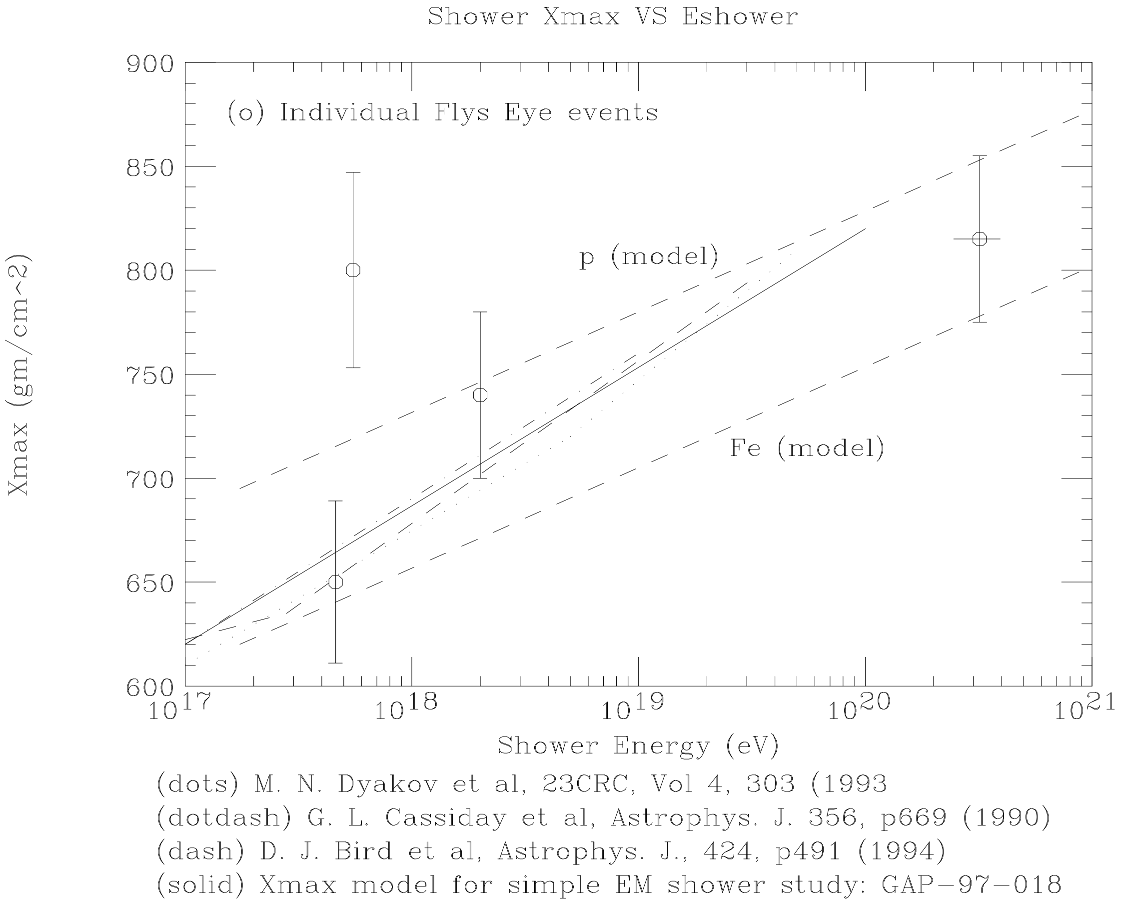
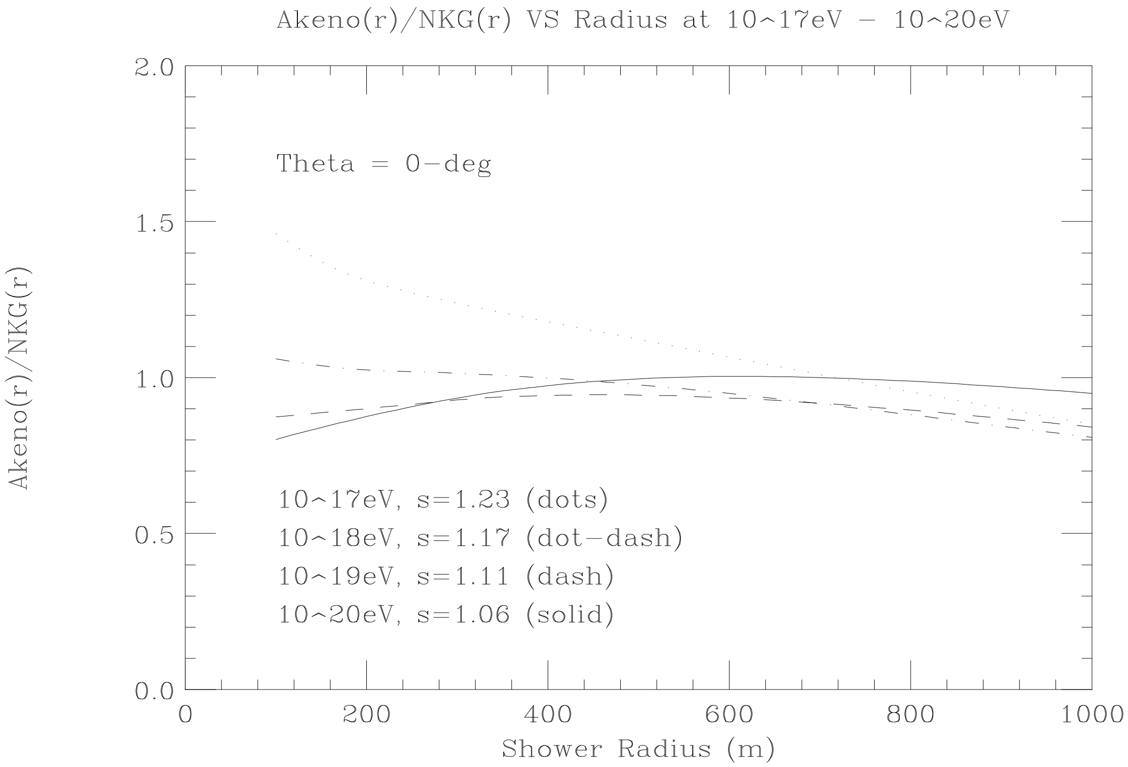


Fig. 1: Parameterization of X_{max} versus E_{shower} used in this study.

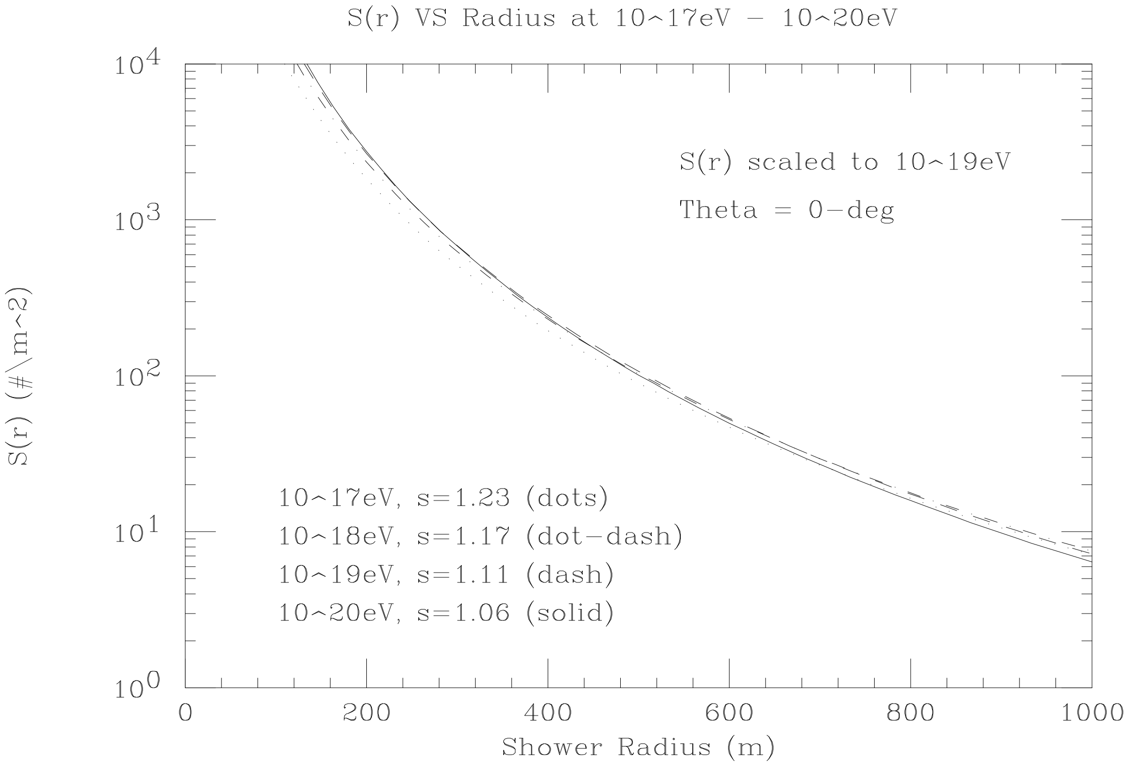


AGASA: S. Yoshida et al, J. Phys. G, Vol 20, 651 (1994)

EM shower model: A. M. Hillas, J. Phys. G, Vol 8, 1461 (1982)

Xmax model for simple EM shower study

Fig. 2: Ratio of the Akeno transverse particle density distribution divided by the analytic model (labelled NKG) versus radius from the shower core evaluated at the atmospheric depth of the Akeno/AGASA array for showers at 0°. The shower age, s, and energy are given in the legend in the figure.

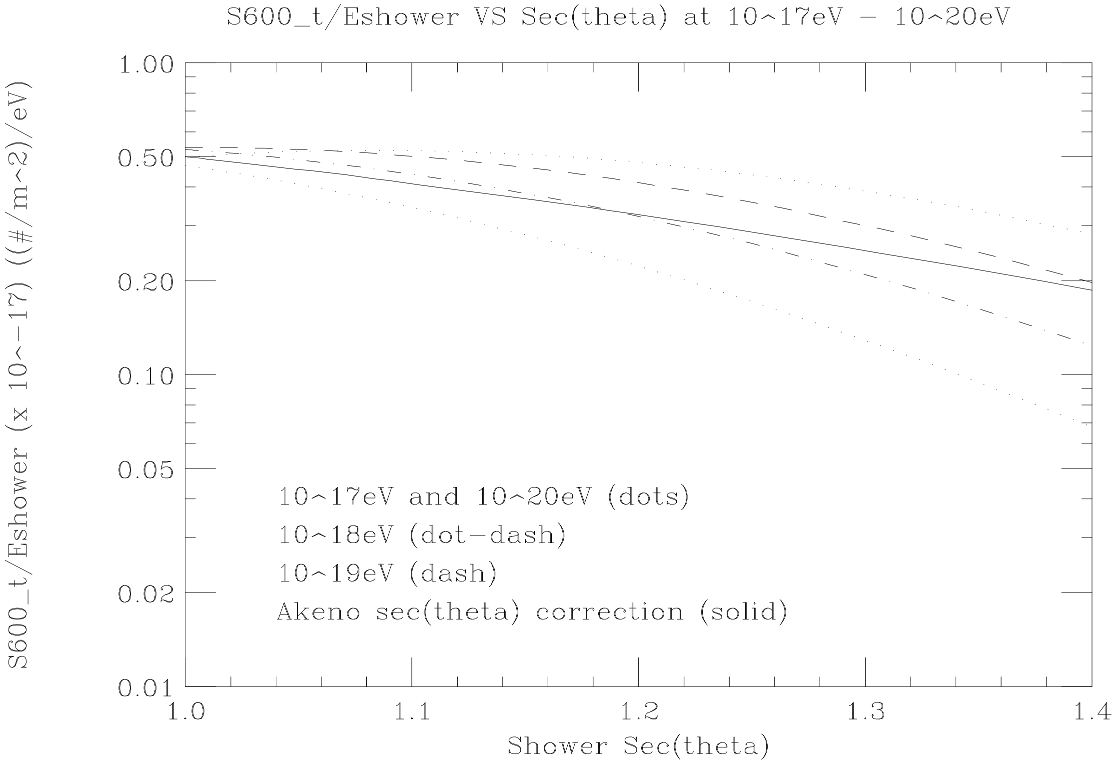


EM shower model: A. M. Hillas, J. Phys. G, Vol 8, 1461 (1982)

EM trans. distrib.: K. Greisen, Ann. Rev. Nucl. Sc., 10, 63 (1960)

Xmax model for simple EM shower study

Fig. 3: The analytic model transverse particle density distributions versus radial distance from the shower core evaluated at the atmospheric depth of the Akeno/AGASA array for showers at 0°. The densities have been scaled by shower energy to 10^{19}eV. The shower age, s, and energy are given in the legend in the figure.



AGASA: S. Yoshida et al, J. Phys. G, Vol 20, 651 (1994)

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Fig. 4: Plot of $S_\theta(600)/E_{shower}$ from the analytic model versus secant of the shower zenith angle evaluated at the atmospheric depth of the Akeno/AGASA array. The Akeno/AGASA zenith angle correction factor is given by the solid line.

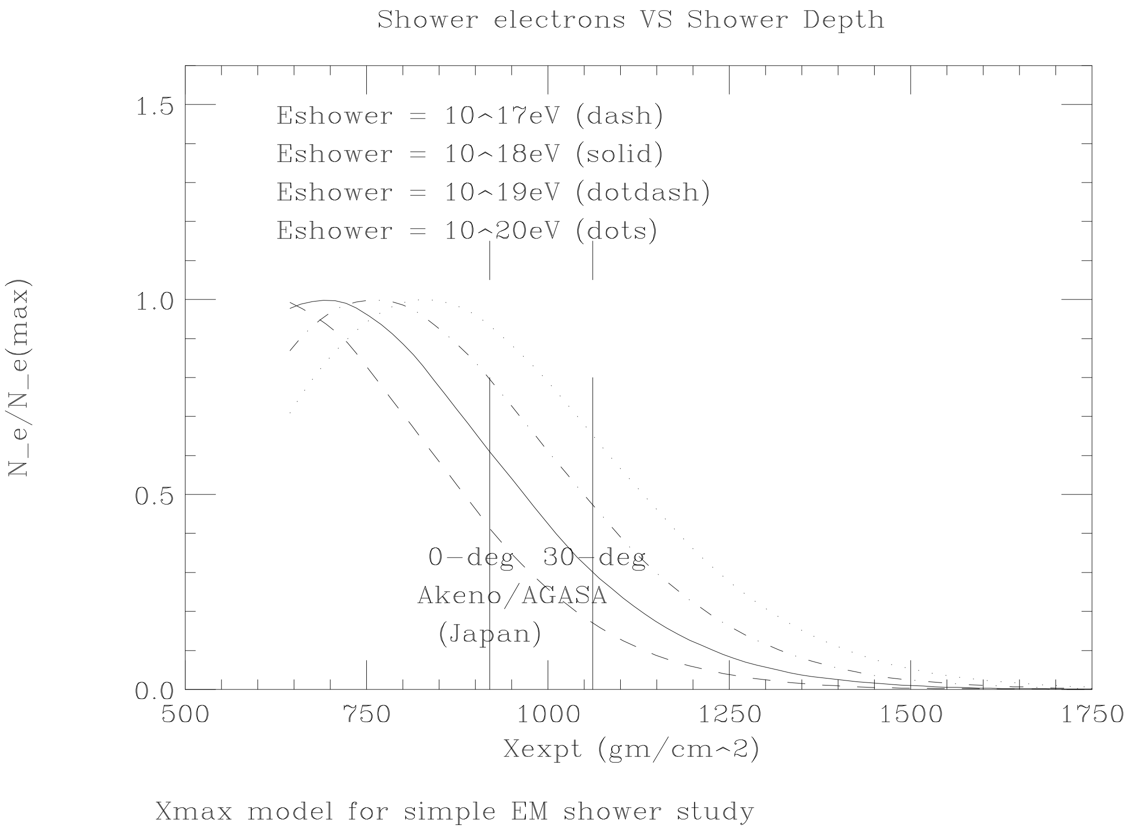


Fig. 5: The longitudinal profile for the number of charged particles for $10^{17}eV \sim E_{shower} \sim 10^{20}eV$. Each curve is independently normalized to a maximum value of 1.

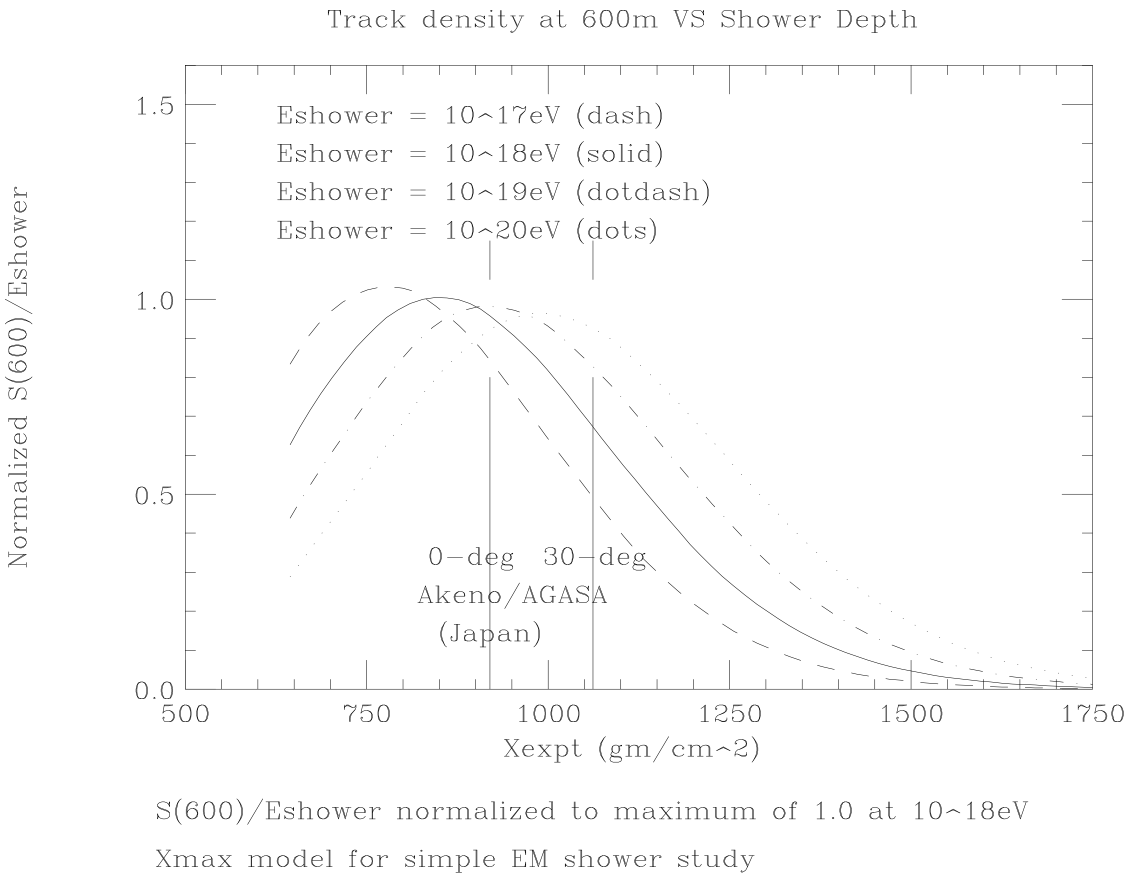


Fig. 6: The longitudinal profile for normalized charged particle density at 600m, $S(600)/E_{shower}$, for $10^{17}eV \sim E_{shower} \sim 10^{20}eV$, versus shower depth. The curves are normalized to a maximum value of 1.0 at 10^{18}eV.

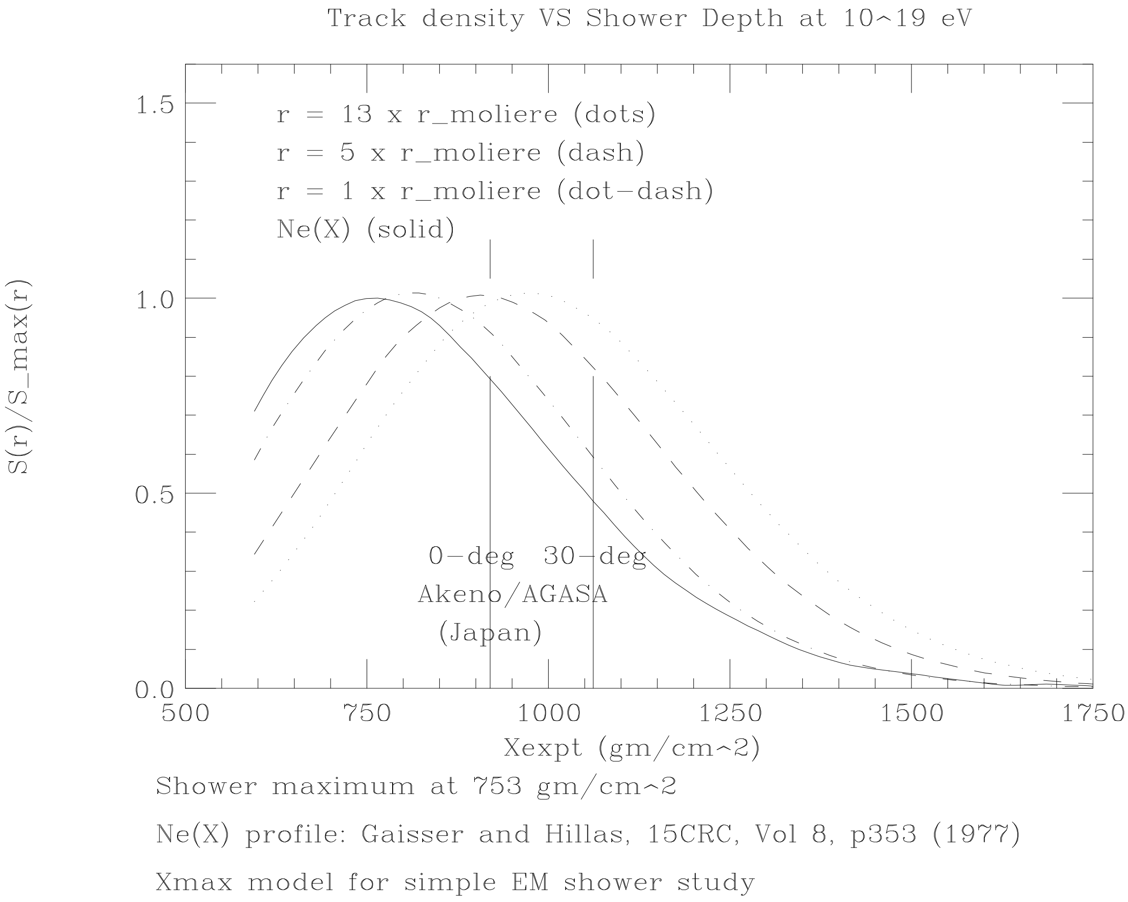


Fig. 7: The longitudinal profile for charged particle density at various radial distances from the shower core versus shower depth for $E_{shower} = 10^{19}$eV. The curves are normalized to a maximum value of 1.0.

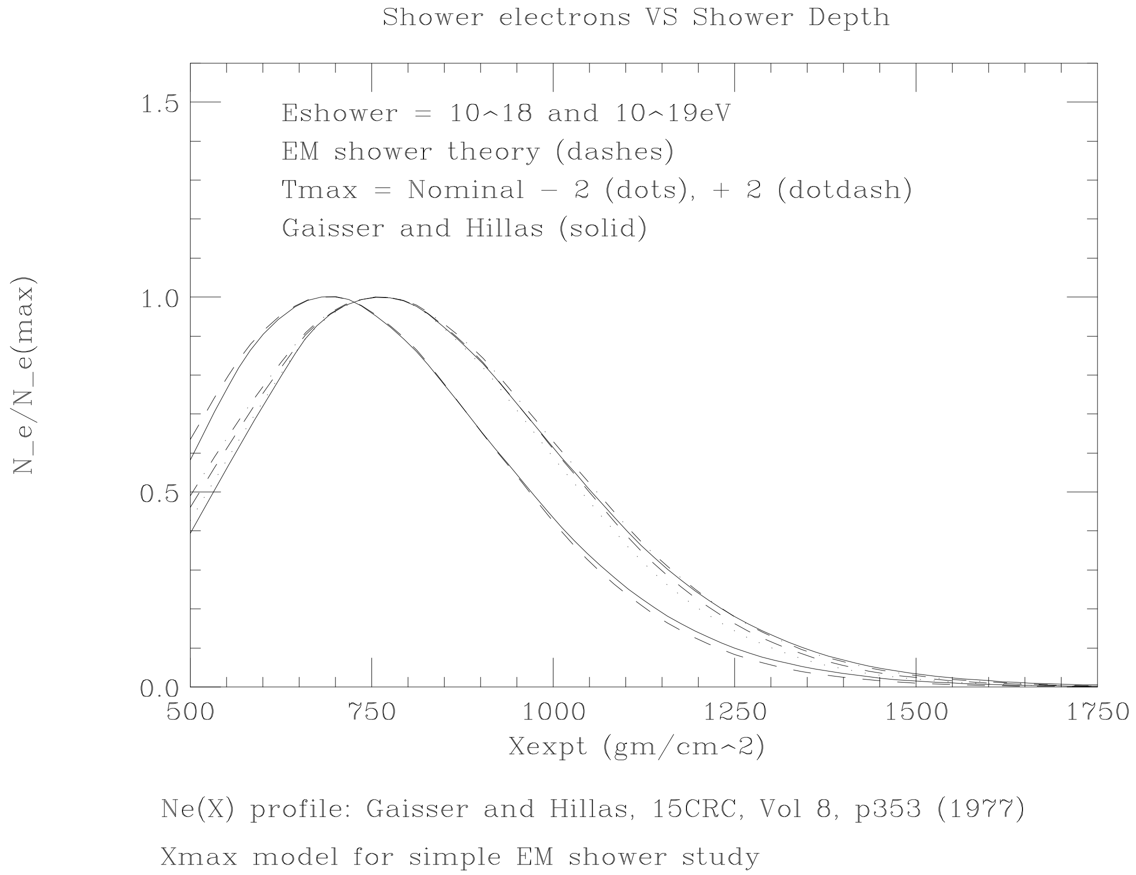
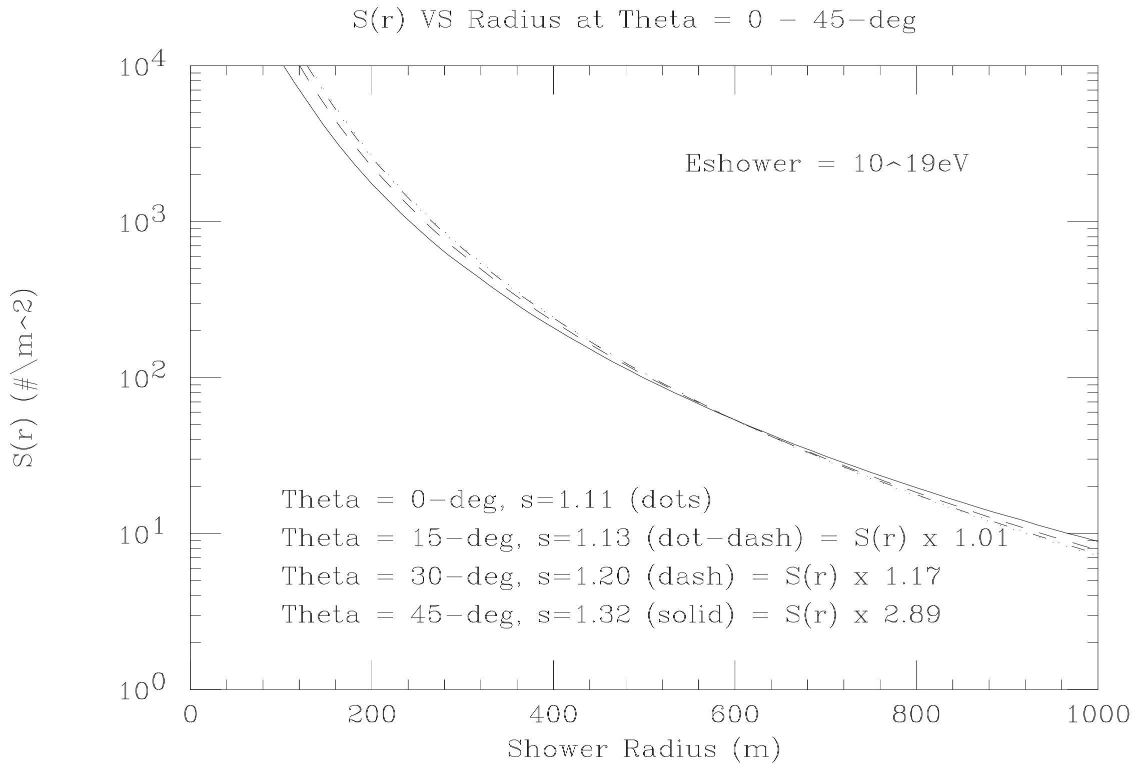


Fig. 8: The longitudinal profile for the total number of charged particles, N_e, is shown for the Gaisser-Hillas parameterization and for the analytic model as a function of shower depth. Predictions are given at 10^{18}eV and 10^{19}eV. At 10^{19}eV results for the analytic model include changing the value of T_{eff}^{max} by ±2 from the nominal value.



S(r) VS Radius at Theta = 0 – 45–deg

EM shower model: A. M. Hillas, J. Phys. G, Vol 8, 1461 (1982)

EM trans. distrib.: K. Greisen, Ann. Rev. Nucl. Sc., 10, 63 (1960)

Xmax model for simple EM shower study

Fig. 9: The analytic model transverse particle density distributions versus radius from shower core evaluated at the atmospheric depth of the Akeno/AGASA array for 10^{19}eV showers. The densities at different zenith angles are scaled to agree at $r = 600$m. The shower age, s, and and zenith angle are given in the legend in the figure.